

Algebraic description of external and internal attributes of fundamental fermions

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Abstract. To describe external and internal attributes of fundamental fermions, a theory of multi-spinor fields is developed on an algebra, a *triplet algebra*, which consists of all the triple-direct-products of Dirac γ -matrices. The triplet algebra is decomposed into the product of two subalgebras, an external algebra and an internal algebra, which are exclusively related with external and internal characteristic of the multi-spinor field named *triplet fields*. All elements of the external algebra which is isomorphic to the original Dirac algebra A_γ are invariant under the action of permutation group S_3 which works to exchange the order of the A_γ elements in the triple-direct-product. The internal algebra is decomposed into the product of two 4^2 dimensional algebras, called the family and color algebras, which describe the family and color degrees of freedom. The family and color algebras have fine substructures with “trio plus solo” ($3+1$) conformations which are irreducible under the action of S_3 . The triplet field has trio plus solo family modes with ordinary tricolor quark and colorless solo lepton components. To incorporate the Weinberg-Salam mechanism, it is required to introduce two types of triplet fields, a left-handed doublet and right-handed singlets of electroweak iso-spin. It is possible to qualify the Yukawa interaction and to make a new interpretation of its coupling constants naturally in an intrinsic mechanism of the triplet field formalism. The ordinary Higgs mechanism leads to a new type of the Dirac mass matrices which can explain all data of quark sector within experimental accuracy.

1. Introduction

Fundamental fermions, quarks and leptons, exist in repetitional chiral modes with color and electroweak symmetries. To describe this rich structure as a whole, it is necessary to prepare a flexible theoretical framework with large capacity. In the present article, a theory of multi-spinor fields is developed on an algebra, a *triplet algebra*, which consists of all the triple-direct-products (TDPs) of Dirac γ -matrices. The triplet algebra is decomposed into the product of two subalgebras, an external algebra and an internal algebra, which are exclusively related with external and internal characteristic of fundamental fermions. The multi-spinor field, which is called a *triplet field*, forms reducible representation of the Lorentz group including spin $\frac{1}{2}$ component fields with a variety of internal attribute and symmetry.

All elements of the external algebra which is isomorphic to the original Dirac algebra A_γ are invariant under the action of permutation group S_3 which works to exchange the order of the A_γ elements in the TDP. The internal algebra is decomposed into the product of two 4^2 dimensional subalgebras, called the family and color algebras, which describe the family and

color degrees of freedom, respectively. Correspondingly, the triplet field possesses independent component fields of maximally quartet family modes and quartet color modes.

We impose the conditions that the family and color algebras have fine substructures which are irreducible under the action of the permutation group S_3 . This condition works to degrade the family and color algebras which are responsible for the quartet structures into fine algebras with “trio plus solo” ($3 + 1$) conformations. As a result, we are able to interpret that the component fields of the triplet field are composed of the ordinary three families and an additional fourth family of the tricolor quark and colorless lepton modes.

To reproduce all the successful results of the standard model and go beyond it, we introduce two types of triplet fields, a left-handed doublet and right-handed singlets of electroweak iso-spin, and apply the Weinberg-Salam theory originally formulated for the electron and its associated neutrino to these triplet fields. The ordinary gauge fields and Higgs field of electroweak $SU(2)$ symmetry are assumed to interact with these triplet fields, and the gauge interactions of color $SU(3)$ symmetry (“baryon minus lepton” $B - L$ charge) are set to take effects between internal quark modes (for quark and lepton modes) of the triplet fields. Thanks to the “trio plus solo” scheme for the family structure, the Yukawa interactions are block diagonalized so that ordinary trio quarks and trio leptons interact among themselves. The solo fermions in the fourth family exist independently of the ordinary trio fermions without observable mixing effects.

The Lagrangian density of the fermion sector is constructed with the triplet fields. By substituting the mode decompositions of the triplet fields, we express the density in terms of component fields of quarks and leptons. One superior aspect of this formalism is calculability of the Yukawa coupling constants as matrix elements between different family modes. It is possible to qualify the Yukawa interaction and to make a new interpretation of its coupling constants with CP-violating phase naturally in an intrinsic mechanism of the triplet field formalism. The ordinary Higgs mechanism results in new and unique Dirac mass matrices which can explain all of the experimental data of the quark mass spectra and weak mixing matrix.

We construct explicitly the triplet algebra and its subalgebra for the external spacetime symmetry and internal attributes of fundamental fermions in §2. Characteristic of the subalgebra for internal properties of fundamental fermions is closely examined in §3 and §4. First, chirality operators acting on the triplet field are decomposed into the sums of the projection operators which can be interpreted to discriminate repetitional degrees of freedom for family structure. Finding the elements which can describe the mixings between the interaction and mass-eigenstate modes, we construct the family algebra with the trio plus solo conformation. Second, the algebra for color symmetry is determined in the set of the elements which are commutative both with the chirality operators and all elements of the algebra for external symmetry. We introduce the triplet field forming a multiple representation of the Lorentz group in §5. In the present scheme, the triplet field has no degrees of freedom for the symmetry of electroweak interaction. In §6, we reformulate the standard model by introducing the set of chiral triplet fields of the left-handed doublet and the right-handed singlets of the Weinberg-Salam symmetry. Explicit form of the Yukawa interaction is given and the Dirac mass matrix with new unique structure is derived. Possible implications of the triplet field formalism for and beyond the standard model are discussed in §7.

2. Triplet algebra and subalgebras for external and internal attributes

The Dirac’s γ -matrices γ_μ ($\mu = 0, 1, 2, 3$) satisfying the Clifford anticommutation relations $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}1$ with $(\eta_{\mu\nu}) = \text{diagonal}(1, -1, -1, -1)$ are introduced solely as the mathematical quantities which carry no direct physical meaning. Hermite conjugate of γ_μ is defined by $\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0$. The γ -matrices generate the 16 dimensional algebra, named a *Dirac algebra*, as

$$A_\gamma = \langle \gamma_\mu \rangle = \{1, \gamma_\mu, \sigma_{\mu\nu}, \gamma_5 \gamma_\mu, \gamma_5\} \quad (1)$$

where $\sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \gamma^5$.

Let us call the TDP of the bases $1, \gamma_\mu, \sigma_{\mu\nu}, \gamma_5\gamma_\mu$ and γ_5 of A_γ a *primitive triplet* and the TDP of arbitrary elements of A_γ a *triplet*. Then, the *triplet algebra* A_T is defined as the 16^3 dimensional algebra spanned by all the linear combinations of triplets. In other words, the triplet algebra A_T is generated in terms of the 12 primitive triplets $\gamma_\mu \otimes 1 \otimes 1, 1 \otimes \gamma_\mu \otimes 1$ and $1 \otimes 1 \otimes \gamma_\mu$ as follows:

$$A_T = \langle \gamma_\mu \otimes 1 \otimes 1, 1 \otimes \gamma_\mu \otimes 1, 1 \otimes 1 \otimes \gamma_\mu \rangle. \quad (2)$$

The transpose (Hermite conjugate) of the primitive triplet is defined by the TDP of its transposed (Hermite conjugate) components of A_γ and the trace of the triplet is set to be the product of the traces of its A_γ components.

The triplet algebra A_T is large enough to form a variety of realizations of the Lorentz group representation. To obtain such kind of multiple spinor representation that accomodates solely the spin $\frac{1}{2}$ states, we note that the primitive triplets

$$\Gamma_\mu = \gamma_\mu \otimes \gamma_\mu \otimes \gamma_\mu \quad (\mu = 0, 1, 2, 3) \quad (3)$$

satisfy the Clifford relations [1, 2]

$$\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\eta_{\mu\nu} I \quad (4)$$

where $I = 1 \otimes 1 \otimes 1$. Hermite conjugate of Γ_μ is defined by $\Gamma_\mu^\dagger = \Gamma_0 \Gamma_\mu \Gamma_0$ and the trace is calculated as $\text{Tr } \Gamma_\mu = 0$ and $\text{Tr } (\Gamma_\mu \Gamma_\nu) = 64\eta_{\mu\nu}$. With these new elements, we are now able to construct a subalgebra for external attributes of the triplet field, the *external algebra* A_{ex} , as follows:

$$A_{ex} = \langle \Gamma_\mu \rangle = \{1, \Gamma_\mu, \Sigma_{\mu\nu}, \Gamma_5 \Gamma_\mu, \Gamma_5\} \quad (5)$$

where

$$\Sigma_{\mu\nu} = -\frac{i}{2}(\Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu) = \sigma_{\mu\nu} \otimes \sigma_{\mu\nu} \otimes \sigma_{\mu\nu} \quad (6)$$

and

$$\Gamma_5 = -i\Gamma_0\Gamma_1\Gamma_2\Gamma_3 = \Gamma^5 = \gamma_5 \otimes \gamma_5 \otimes \gamma_5. \quad (7)$$

Evidently, A_{ex} is isomorphic to the original Dirac algebra A_γ .

It is readily proved that the operators

$$M_{\mu\nu} = \frac{1}{2}\Sigma_{\mu\nu} \quad (8)$$

are subject to the commutation relations of the Lie algebra for the orthogonal group $O(1,3)$ and $M_{\mu\nu}$ and Γ_λ satisfy the relations $[M_{\mu\nu}, \Gamma_\lambda] = i\eta_{\lambda\nu}\Gamma_\mu - i\eta_{\lambda\mu}\Gamma_\nu$. Accordingly, it is possible to postulate that the operators $M_{\mu\nu}$ generate the Lorentz transformations in our 4 dimensional spacetime and that the subscripts of operators consisting of Γ_μ refer directly to the superscripts of the spacetime coordinates $\{x^\mu\}$ of the 4 dimensional world where we exist as observers. Therefore, it is naturally defined that the raising and lowering operations of index of Γ_μ by the metric $\eta_{\mu\nu}$ as $\Gamma^\mu = \eta^{\mu\nu}\Gamma_\nu$ and $\Gamma_\mu = \eta_{\mu\nu}\Gamma^\nu$.

The internal attributes of fundamental fermions should be fixed independently of the inertial frame of reference in which observations are made. This means that the generators specifying their internal attributes must be commutative with the elements of the subalgebra, $\{\Sigma_{\mu\nu}\}$, for the Lorentz transformations. Accordingly, we are led naturally to postulate that the *internal algebra*, A_{in} , is defined by the *centralizer* of $\{\Sigma_{\mu\nu}\}$ of the algebra for the Lorentz transformation as follows:

$$A_{in} = C(\{\Sigma_{\mu\nu}\} : A_T) = \{X \in A_T : [X, \Sigma_{\mu\nu}] = 0\}. \quad (9)$$

The primitive triplet of this centralizer is proved to consist of either an even number or an odd number of component γ_μ -matrix for all μ [1]. Consequently, the internal algebra is determined to be

$$A_{in} = \langle 1 \otimes \gamma_\mu \otimes \gamma_\mu, \gamma_\mu \otimes 1 \otimes \gamma_\mu, \gamma_5 \otimes \gamma_5 \otimes \gamma_5 \rangle. \quad (10)$$

Now, the triplet algebra is shown to have the decomposition

$$A_T = A_{ex}A_{in}, \quad A_{ex} \cap A_{in} = \{\Gamma_5\}. \quad (11)$$

The element Γ_5 belongs not only to the external algebra but also to the internal algebra. This is a crucial aspect of the present theory which makes it possible to realize the repetitional family structure as an internal attribute of the triplet field.

As is evident from the construction so far, the component γ -matrices carry no direct physical meaning and regarded as playing only the role of *alphabets*. It is selected elements of the triplet algebra A_T that play the role of *codons* carrying the direct physical meanings.

3. Subdivision of chirality operators and repetitional modes of family structure

Our next task is to examine detailed structures of the internal algebra A_{in} in (10) and to inquire what sorts of internal attributes of the fundamental fermions can be inscribed on the algebra. For its purpose, we should notice the point that complexities in flavor physics rise from mixture of freedom of families and chiralities.

It is straightforward to verify that the chirality operators

$$L = \frac{1}{2}(I - \Gamma_5), \quad R = \frac{1}{2}(I + \Gamma_5) \quad (12)$$

have the following subdivisions in the internal algebra A_{in} as

$$L = \ell \otimes r \otimes r + r \otimes \ell \otimes r + r \otimes r \otimes \ell + \ell \otimes \ell \otimes \ell \quad (13)$$

and

$$R = r \otimes \ell \otimes \ell + \ell \otimes r \otimes \ell + \ell \otimes \ell \otimes r + r \otimes r \otimes r \quad (14)$$

where

$$\ell = \frac{1}{2}(1 - \gamma_5), \quad r = \frac{1}{2}(1 + \gamma_5). \quad (15)$$

Note that this subdivision cannot be realized in the algebra A_{ex} .

To give independent physical roles and meanings to the subdivided terms in (13) and (14), let us introduce the following operators in A_{in} as

$$\left\{ \begin{array}{ll} \Pi_{1L} = \ell \otimes r \otimes r, & \Pi_{1R} = r \otimes \ell \otimes \ell, \\ \Pi_{2L} = r \otimes \ell \otimes r, & \Pi_{2R} = \ell \otimes r \otimes \ell, \\ \Pi_{3L} = r \otimes r \otimes \ell, & \Pi_{3R} = \ell \otimes \ell \otimes r, \\ \Pi_{4L} = \ell \otimes \ell \otimes \ell, & \Pi_{4R} = r \otimes r \otimes r, \end{array} \right. \quad (16)$$

which are chiral projection operators obeying the relations

$$\Pi_{ih}\Pi_{jh'} = \delta_{ij}\delta_{hh'}\Pi_{ih}, \quad \sum_{ih} \Pi_{ih} = I \quad (17)$$

for $i, j = 1, 2, 3, 4$ and $h, h' = L, R$. Next, we define the non-chiral projection operators

$$\Pi_i = \Pi_{iL} + \Pi_{iR} \quad (18)$$

satisfying the relations

$$\Pi_i \Pi_j = \delta_{ij} \Pi_i, \quad \sum_i \Pi_i = I. \quad (19)$$

Note that the chiral projection operators can be reproduced from the non-chiral projection operators as

$$\Pi_{iL} = L \Pi_i, \quad \Pi_{iR} = R \Pi_i. \quad (20)$$

Here we hypothesize that this decomposition of the chirality operators explains the origin of the family structure of fundamental fermions. There are maximally four family degrees of freedom in this scheme. However, experimental data have established now that the family number of ordinary quarks and leptons are three in low energy regime of flavor physics. To incorporate this observed characteristics, let us notice the different behaviors of the projection operators Π_i under actions of the discrete group S_3 which works to exchange the order of the elements of A_γ in the triple-direct product. Under the action of this group, the three operators Π_i ($i = 1, 2, 3$) are changed with each other and the operator Π_4 remains invariant. It is natural and relevant to make use of this feature to divide the quartet into “trio and solo” sectors for the family structure.

The projection operators to the trio and solo sectors are constructed in the forms

$$\Pi_{(t)} = \Pi_1 + \Pi_2 + \Pi_3 = \frac{1}{4} (3I - 1 \otimes \gamma_5 \otimes \gamma_5 - \gamma_5 \otimes 1 \otimes \gamma_5 - \gamma_5 \otimes \gamma_5 \otimes 1) \quad (21)$$

and

$$\Pi_{(s)} = \Pi_4 = \frac{1}{4} (I + 1 \otimes \gamma_5 \otimes \gamma_5 + \gamma_5 \otimes 1 \otimes \gamma_5 + \gamma_5 \otimes \gamma_5 \otimes 1) \quad (22)$$

which are subject to the relations $\Pi_{(a)} \Pi_{(b)} = \delta_{ab} \Pi_{(a)}$ for $a, b = t, s$. We postulate here that the ordinary quarks and leptons belong to the trio sector and a fourth family of additional quark and lepton forms the solo sector.

Quarks and leptons can exist in either modes of the interaction and the mass eigenstates. To allow transmutations among two modes, it is necessary to secure a sufficient number of ingredients in the algebra for the trio sector, *trio algebra*. For its purpose, let us introduce the three elements of A_γ as

$$\sigma_1 = \gamma_0, \quad \sigma_2 = i\gamma_0 \gamma_5, \quad \sigma_3 = \gamma_5 \quad (23)$$

which are subject to the multiplication rules of the Pauli algebra, *i.e.*, $\sigma_a \sigma_b = \delta_{ab} I + i\epsilon_{abc} \sigma_c$. Taking the TDPs of these elements, we are able to construct the bases for the trio algebra as follows:

$$\left\{ \begin{array}{l} \pi_1 = \frac{1}{2} (\sigma_1 \otimes \sigma_1 \otimes 1 + \sigma_2 \otimes \sigma_2 \otimes 1), \\ \pi_2 = \frac{1}{2} (\sigma_1 \otimes \sigma_2 \otimes \sigma_3 - \sigma_2 \otimes \sigma_1 \otimes \sigma_3), \\ \pi_3 = \frac{1}{2} (1 \otimes \sigma_3 \otimes \sigma_3 - \sigma_3 \otimes 1 \otimes \sigma_3), \\ \pi_4 = \frac{1}{2} (\sigma_1 \otimes 1 \otimes \sigma_1 + \sigma_2 \otimes 1 \otimes \sigma_2), \\ \pi_5 = \frac{1}{2} (\sigma_1 \otimes \sigma_3 \otimes \sigma_2 - \sigma_2 \otimes \sigma_3 \otimes \sigma_1), \\ \pi_6 = \frac{1}{2} (1 \otimes \sigma_1 \otimes \sigma_1 + 1 \otimes \sigma_2 \otimes \sigma_2), \\ \pi_7 = \frac{1}{2} (\sigma_3 \otimes \sigma_1 \otimes \sigma_2 - \sigma_3 \otimes \sigma_2 \otimes \sigma_1), \\ \pi_8 = \frac{1}{2\sqrt{3}} (1 \otimes \sigma_3 \otimes \sigma_3 + \sigma_3 \otimes 1 \otimes \sigma_3 - 2\sigma_3 \otimes \sigma_3 \otimes 1). \end{array} \right. \quad (24)$$

It is somehow laborious but straightforward to verify that these eight operators belonging to A_{in} satisfy the commutation relations

$$[\pi_j, \pi_k] = 2f_{jkl}\pi_l \quad (25)$$

and the anticommutation relations

$$\{\pi_j, \pi_k\} = \frac{4}{3}\delta_{jk}\Pi_{(t)} + 2d_{jkl}\pi_l \quad (26)$$

of the Lie algebra $\mathfrak{su}(3)$, where f_{jkl} and d_{jkl} are the symmetric and antisymmetric structure constants characterizing the algebra. The operators π_j are self-adjoint and have the traces

$$\text{Tr } \pi_j = 0, \quad \text{Tr } \pi_j \pi_k = 32\delta_{jk}. \quad (27)$$

The anticommutation relation (26) shows that the projection operator $\Pi_{(t)}$ is generated from the operators π_j . The identities

$$\Pi_{(a)} \pi_j = \delta_{aj} \pi_j \quad (28)$$

hold for $a = t, s$ and $j = 1, 2, \dots, 8$. Note that $\pi_j (j = 1, 2, \dots, 8)$ are simultaneous eigen-operators of $\Pi_{(t)}$ and $\Pi_{(s)}$ with respective eigenvalues 1 and 0. The projection operator $\Pi_{(s)}$ is orthogonal to all elements of the trio algebra. Therefore, it is possible to afford a role to discriminate the trio and solo sectors to the operator

$$Q_X = \Pi_{(s)} \quad (29)$$

which has the eigenvalue 0 and 1.

Now we are able to define the subalgebras, *trio algebra* $A_{(t)}$ and *solo algebra* $A_{(s)}$, which can specify the trio and solo degrees of freedom by

$$A_{(t)} = \{ \Pi_{(t)}; \pi_1, \pi_2, \dots, \pi_8 \} \quad (30)$$

and

$$A_{(s)} = \{ Q_X \}. \quad (31)$$

With these algebras and the algebra of the chiral projection operator $\{L, R\}$, an algebra for the repetitional family structure, *family algebra* is constructed as follows:

$$A_f = \{L, R\}\{A_{(t)} \cup A_{(s)}\} \subset A_{in}, \quad A_{(t)} \cap A_{(s)} = \emptyset. \quad (32)$$

It is laborious but unmistakable to prove that the identities

$$\pi_j A_{(a)} = \delta_{aj} A_{(t)} \quad (33)$$

hold for $a = t, s$ and $j = 1, 2, \dots, 8$. Namely, $A_{(t)}$ and $A_{(s)}$ are simultaneous eigen-algebras of the operators π_j with respective eigenvalues 1 and 0. We can also verify that both algebras are irreducible under the action of the permutation group S_3 .

4. Color symmetry and B-L charge

In the previous section, the algebra for family structure A_f with the trio and solo substructures is extracted from the internal algebra A_{in} . Next, it is necessary to investigate the structure of the centralizer $C(A_f : A_{in})$ of A_f in A_{in} .

Let us select out the following three elements from the Dirac algebra A_γ as

$$\rho_1 = i\gamma_2\gamma_3, \quad \rho_2 = i\gamma_3\gamma_1, \quad \rho_3 = i\gamma_1\gamma_2 \quad (34)$$

which are commutative with σ_a ($a = 1, 2, 3$) in (23) and satisfy the multiplication rules of the Pauli algebra, *i.e.*, $\rho_a\rho_b = \delta_{ab}I + i\epsilon_{abc}\rho_c$. In terms of the TDPs of these elements, we are able to construct explicitly the centralizer $C(A_f : A_{in})$.

The basic algebras $\{\sigma_a\}$ in (23) and $\{\rho_a\}$ in (34) are isomorphic with each other. Therefore, sets of their TDPs can share analogous structures. It is, however, necessary to taken into account the point that, while $\gamma_5 \otimes \gamma_5 \otimes \gamma_5 = -\Gamma_5$ belongs to A_{in} , the operator $\rho_3 \otimes \rho_3 \otimes \rho_3 = -\Sigma_{12}$ is not included in A_{in} . This means that no counter parts of the chiral operators can exist in the centralizer $C(A_f : A_{in})$. We must construct the TDPs of $\{\rho_a\}$ in analogy with the non-chiral operators Π_i , $\Pi_{(a)}$ and π_i to form $C(A_f : A_{in})$.

In accord with the operators Π_i and $\Pi_{(a)}$, the centralizer can possess maximally four operators which have the ‘‘trio plus solo’’ conformation. Here we postulate that those operators carry the roles to describe the color degrees of freedom for the tricolor quark and colorless lepton states which are established firmly in flavor physics.

In parallel with (21) and (22), it is possible to introduce the following projection operators for the trio quark states and the solo lepton state as

$$\Lambda^{(q)} = \frac{1}{4}(3I - 1 \otimes \rho_3 \otimes \rho_3 - \rho_3 \otimes 1 \otimes \rho_3 - \rho_3 \otimes \rho_3 \otimes 1) \quad (35)$$

and

$$\Lambda^{(\ell)} = \frac{1}{4}(I + 1 \otimes \rho_3 \otimes \rho_3 + \rho_3 \otimes 1 \otimes \rho_3 + \rho_3 \otimes \rho_3 \otimes 1) \equiv \Lambda_\ell \quad (36)$$

which are subject to the relations

$$\Lambda^{(a)}\Lambda^{(b)} = \delta_{ab}\Lambda^{(a)} \quad (37)$$

where $a, b = q, \ell$. Accordingly, we can construct the charge operator for *baryon number minus lepton number* ($B - L$) in the form

$$Q_{B-L} = \frac{1}{3}\Lambda^{(q)} - \Lambda^{(\ell)} = -\frac{1}{3}(1 \otimes \rho_3 \otimes \rho_3 + \rho_3 \otimes 1 \otimes \rho_3 + \rho_3 \otimes \rho_3 \otimes 1) \quad (38)$$

which has eigenvalues $1/3$ and -1 . It is possible to subdivide the operator $\Lambda^{(q)}$ symmetrically into

$$\Lambda^{(q)} = \Lambda_r + \Lambda_y + \Lambda_g \quad (39)$$

where

$$\begin{cases} \Lambda_r = \frac{1}{4}(I + 1 \otimes \rho_3 \otimes \rho_3 - \rho_3 \otimes 1 \otimes \rho_3 - \rho_3 \otimes \rho_3 \otimes 1), \\ \Lambda_y = \frac{1}{4}(I - 1 \otimes \rho_3 \otimes \rho_3 + \rho_3 \otimes 1 \otimes \rho_3 - \rho_3 \otimes \rho_3 \otimes 1), \\ \Lambda_g = \frac{1}{4}(I - 1 \otimes \rho_3 \otimes \rho_3 - \rho_3 \otimes 1 \otimes \rho_3 + \rho_3 \otimes \rho_3 \otimes 1) \end{cases} \quad (40)$$

can be interpreted as the projection operators into tricolor quark modes. The four operators Λ_a satisfy the relations

$$\Lambda_a\Lambda_b = \delta_{ab}\Lambda_a, \quad \sum_a \Lambda_a = I \quad (41)$$

for $a, b = r, y, g, \ell$.

Here again, taking resort to the isomorphism of the basic algebras $\{\sigma_a\}$ in (23) and $\{\rho_a\}$ in (34), we construct the generators of transformations among the tricolor quark states in analogy with (24) as follows:

$$\left\{ \begin{array}{l} \lambda_1 = \frac{1}{2} (\rho_1 \otimes \rho_1 \otimes 1 + \rho_2 \otimes \rho_2 \otimes 1), \\ \lambda_2 = \frac{1}{2} (\rho_1 \otimes \rho_2 \otimes \rho_3 - \rho_2 \otimes \rho_1 \otimes \rho_3), \\ \lambda_3 = \frac{1}{2} (1 \otimes \rho_3 \otimes \rho_3 - \rho_3 \otimes 1 \otimes \rho_3), \\ \lambda_4 = \frac{1}{2} (\rho_1 \otimes 1 \otimes \rho_1 + \rho_2 \otimes 1 \otimes \rho_2), \\ \lambda_5 = \frac{1}{2} (\rho_1 \otimes \rho_3 \otimes \rho_2 - \rho_2 \otimes \rho_3 \otimes \rho_1), \\ \lambda_6 = \frac{1}{2} (1 \otimes \rho_1 \otimes \rho_1 + 1 \otimes \rho_2 \otimes \rho_2), \\ \lambda_7 = \frac{1}{2} (\rho_3 \otimes \rho_1 \otimes \rho_2 - \rho_3 \otimes \rho_2 \otimes \rho_1), \\ \lambda_8 = \frac{1}{2\sqrt{3}} (1 \otimes \rho_3 \otimes \rho_3 + \rho_3 \otimes 1 \otimes \rho_3 - 2\rho_3 \otimes \rho_3 \otimes 1) \end{array} \right. \quad (42)$$

which satisfy the commutation relations

$$[\lambda_j, \lambda_k] = 2f_{jkl}\lambda_l \quad (43)$$

and the anticommutation relations

$$\{\lambda_j, \lambda_k\} = \frac{4}{3}\delta_{jk}\Lambda^{(q)} + 2d_{jkl}\lambda_l \quad (44)$$

of the Lie algebra $\mathfrak{su}(3)$.

The operators λ_j are self-adjoint and have the traces

$$\text{Tr}\lambda_j = 0, \quad \text{Tr}\lambda_j\lambda_k = 32\delta_{jk}. \quad (45)$$

Corresponding to (28), the operators $\Pi^{(a)}$ and π_j satisfy the identities

$$\Lambda^{(a)}\lambda_j = \delta_{aq}\lambda_j \quad (46)$$

which hold for $a = q, \ell$ and $j = 1, 2, \dots, 8$. Evidently, the operators λ_j works to annihilate the leptonic mode and the operator $\Lambda^{(\ell)}$ is orthogonal to all elements λ_j of the trio quark algebra. Now we are able to define the subalgebras, *trio quark algebra* $A^{(q)}$ and *solo lepton algebra* $A^{(\ell)}$, which can specify the trio and solo degrees of freedom by

$$A^{(q)} = \{ \Lambda^{(q)} ; \lambda_1, \lambda_2, \dots, \lambda_8 \} \quad (47)$$

and

$$A^{(\ell)} = \{ \Lambda^{(\ell)} = \Lambda_\ell \}. \quad (48)$$

With these algebras, an algebra for the four color degrees of freedom, *color algebra* is constructed as follows:

$$A_c = A^{(q)} \cup A^{(\ell)} \subset A_{in}, \quad A^{(q)} \cap A^{(\ell)} = \emptyset. \quad (49)$$

Just as (33), the operators λ_j and the algebras $A^{(q)}$ and $A^{(\ell)}$ satisfy the identities

$$\lambda_j A^{(a)} = \delta_{aq} A^{(q)} \quad (50)$$

hold for $a = q, \ell$ and $j = 1, 2, \dots, 8$. Namely, $A^{(q)}$ and $A^{(\ell)}$ are simultaneous eigen-algebras of the operators λ_j . Both algebras are proved to be irreducible under the action of the permutation group S_3 .

5. Triplet field: Multiple spinor representation of the Lorentz group

With these formation of the external and internal algebras, we are now able to introduce the triplet field $\Psi(x)$, on the spacetime point x^μ , which behave like the TDP of four dimensional Dirac spinor. The triplet field which span 4^3 dimensional vector space possesses 4^2 component fields with spin $\frac{1}{2}$ with internal attributes of color and family degrees of freedom.

With the triplet field $\Psi(x)$ and its adjoint field

$$\bar{\Psi}(x) = \Psi^\dagger(x)\Gamma_0, \quad (51)$$

the Lorentz invariant scalar product is defined as

$$\bar{\Psi}(x)\Psi(x) = \sum_{abc} \bar{\Psi}_{abc}(x)\Psi_{abc}(x). \quad (52)$$

Under the proper Lorentz transformation $x'^\mu = \Omega^\mu{}_\nu x^\nu$ where $\Omega_{\lambda\mu}\Omega^\lambda{}_\nu = \eta_{\mu\nu}$ and $\det \Omega = 1$, the triplet field and its adjoint are transformed as

$$\Psi'(x') = S(\Omega)\Psi(x), \quad \bar{\Psi}'(x') = \bar{\Psi}(x)S^{-1}(\Omega) \quad (53)$$

where the transformation matrix is explicitly given by

$$S(\Omega) = \exp\left(-\frac{i}{2}M_{\mu\nu}\omega^{\mu\nu}\right) \quad (54)$$

with the generator for the spacetime rotations in (8) and the angles $\omega^{\mu\nu}$ in the μ - ν planes. For the discrete spacetime transformations such as the space inversion, the time reversal and the charge conjugation, the present scheme retains exactly the same structure as the ordinary Dirac theory.

At the level of the algebra, the family and color degrees of freedom have dual structures: the part $A_{(t)} \cup A_{(s)}$ of the family algebra is isomorphic to the color algebra $A_c = A^{(q)} \cup A^{(\ell)}$. This duality is not realized in the observed characteristics of fundamental fermions. The breakdown of the duality existing in the algebras arises at the level of continuous group formations. Here we introduce the group of color symmetry by

$$SU_c(3) = \left\{ \exp\left(-\frac{i}{2} \sum_j \lambda_j \theta^j(x)\right) \right\} \quad (55)$$

and the group of Q_{B-L} charge and Q_X charge, respectively, by

$$U_{B-L}(1) = \left\{ \exp\left(-\frac{i}{2}Q_{B-L}\theta_{B-L}(x)\right) \right\}, \quad U_X(1) = \left\{ \exp\left(-\frac{i}{2}Q_X\theta_X(x)\right) \right\} \quad (56)$$

where $\theta^j(x)$, $\theta_{B-L}(x)$ and $\theta_X(x)$ are arbitrary real functions of space-time. Under the action of these groups, the bilinear Lorentz invariants in (52) is postulated to be invariant.

The trio plus solo construction of the internal algebra renders the triplet field $\Psi(x)$ to possess the following decompositions as

$$\begin{aligned} \Psi(x) &= \Psi^{(q)}(x) + \Psi^{(\ell)}(x) \\ &= \Psi_{(t)}(x) + \Psi_{(s)}(x) \\ &= \Psi_{(t)}^{(q)}(x) + \Psi_{(s)}^{(q)}(x) + \Psi_{(t)}^{(\ell)}(x) + \Psi_{(s)}^{(\ell)}(x) \end{aligned} \quad (57)$$

with

$$\Psi_{(b)}^{(a)}(x) = \Lambda^{(a)}\Psi_{(b)}(x) = \Pi_{(b)}\Psi^{(a)}(x) = \Lambda^{(a)}\Pi_{(b)}\Psi(x) \quad (58)$$

where $a = q, \ell$ and $b = t, s$. Accordingly, the bilinear Lorentz invariant in (52) is expressed in terms of the separate invariants of the trio and solo sectors as

$$\bar{\Psi}(x)\Psi(x) = \sum_{a=l,\ell} \bar{\Psi}^{(a)}(x)\Psi^{(a)}(x) = \sum_{b=t,s} \bar{\Psi}_{(b)}(x)\Psi_{(b)}(x) = \sum_{a=l,\ell;b=t,s} \bar{\Psi}_{(b)}^{(a)}(x)\Psi_{(b)}^{(a)}(x). \quad (59)$$

To extract the component fields of internal modes of the triplet field, it is convenient to introduce the bracket symbols for the projection operators Λ_a and Π_{ih} by

$$\Lambda_a = |a\rangle\langle a|, \quad \Pi_{ih} = |ih\rangle\langle ih|. \quad (60)$$

For the triplet field and its conjugate, the projection operators work as follows:

$$\Lambda_a\Pi_{ih}\Psi(x) = \Lambda_a|ih\rangle\langle ih|\Psi(x) = |aih\rangle\langle aih|\Psi(x) = |aih\rangle\Psi_{aih}(x) \quad (61)$$

and

$$\bar{\Psi}(x)\Lambda_a\Pi_{ih} = \langle\bar{\Psi}(x)\Lambda_a|ih\rangle\langle ih| = \langle\bar{\Psi}(x)|aih\rangle\langle aih| = \bar{\Psi}_{ai\bar{h}}(x)\langle aih| \quad (62)$$

where $\Psi_{aih}(x)$ and $\bar{\Psi}_{aih}(x)$ are chiral component fields, and \bar{h} implies that $\bar{L} = R$ and $\bar{R} = L$. Then, the decomposition of the bilinear scalar and vector forms of the triplet fields can be achieved as follows:

$$\bar{\Psi}(x)\Psi(x) = \sum_a \sum_{ih} \bar{\Psi}(x)\Lambda_a\Pi_{ih}\Psi(x) = \sum_a \sum_{ih} \bar{\Psi}_{ai\bar{h}}(x)\Psi_{aih}(x) \quad (63)$$

and

$$\bar{\Psi}(x)\Gamma_\mu\Psi(x) = \sum_a \sum_{ih} \bar{\Psi}(x)\Gamma_\mu\Lambda_a\Pi_{ih}\Psi(x) = \sum_a \sum_{ih} \bar{\Psi}_{ai\bar{h}}(x)\Gamma_\mu\Psi_{aih}(x). \quad (64)$$

6. Standard model in the triplet field formalism

To incorporate the Weinberg-Salam mechanism in the present scheme, it is necessary to prepare a set of triplet fields which form representations of the symmetry group $SU_L(2) \times U_y(1)$.

We introduce the left-handed triplet field constituting the doublet of electroweak isospin $\frac{1}{2}\tau_a$ with $y = 0$ as

$$\Psi_L(x) = \begin{pmatrix} \Psi_u(x) \\ \Psi_d(x) \end{pmatrix}_L \quad (65)$$

where $\Psi_u(x)$ and $\Psi_d(x)$ signify the up and down isospin components, respectively, and the right-handed triplet fields forming the electroweak isospin singlets with $y = 1$ and $y = -1$ as follows:

$$\Psi_U(x), \quad \Psi_D(x). \quad (66)$$

The chiral fabric of these triplets claims the conditions: $R\Psi_u = R\Psi_d = 0$ and $L\Psi_U = L\Psi_D = 0$. The breakdown of the electroweak symmetry is assumed to be triggered by the ordinary Higgs field:

$$\phi(x) = \begin{pmatrix} \phi^+(x) \\ \phi^0(x) \end{pmatrix}, \quad \tilde{\phi}(x) = \begin{pmatrix} \phi^{0*}(x) \\ -\phi^-(x) \end{pmatrix}. \quad (67)$$

The quark and lepton parts of the electroweak doublet $\Psi_L(x)$ have the component modes expressed schematically as

$$\Psi^{(q)}(x) = \Lambda^{(q)}\Psi_L(x) = \begin{pmatrix} \Lambda^{(q)}\Lambda_a\Pi_{iL}\Psi_u \\ \Lambda^{(q)}\Lambda_a\Pi_{iL}\Psi_d \end{pmatrix}_L = \begin{pmatrix} u_a & c_a & t_a & \cdot \\ d_a & s_a & b_a & \cdot \end{pmatrix}_L \quad (68)$$

and

$$\Psi^{(\ell)}(x) = \Lambda^{(\ell)}\Psi_L(x) = \begin{pmatrix} \Lambda^{(\ell)}\Pi_{iL}\Psi_u \\ \Lambda^{(\ell)}\Pi_{iL}\Psi_d \end{pmatrix}_L = \begin{pmatrix} \nu_e & \nu_\mu & \nu_\tau & \cdot \\ e & \mu & \tau & \cdot \end{pmatrix}_L \quad (69)$$

where the symbol \cdot expresses the fourth fermions with $Q_X = 1$ charge which remain anonymous at present. Similarly, the electroweak singlets $\Psi_U(x)$ and $\Psi_D(x)$ possess the quark and lepton parts with contents of the component modes as follows:

$$\Psi^{(u)}(x) = \Lambda^{(q)}\Psi_U(x) = \left(\Lambda^{(q)}\Lambda_a\Pi_{iR}\Psi_U\right) = \begin{pmatrix} u_a & c_a & t_a & \cdot \end{pmatrix}_R, \quad (70)$$

$$\Psi^{(d)}(x) = \Lambda^{(q)}\Psi_D(x) = \left(\Lambda^{(q)}\Lambda_a\Pi_{iR}\Psi_D\right) = \begin{pmatrix} d_a & s_a & b_a & \cdot \end{pmatrix}_R, \quad (71)$$

$$\Psi^{(\nu)}(x) = \Lambda^{(\ell)}\Psi_U(x) = \left(\Lambda^{(\ell)}\Pi_{iR}\Psi_U\right) = \begin{pmatrix} \nu_e & \nu_\mu & \nu_\tau & \cdot \end{pmatrix}_R, \quad (72)$$

and

$$\Psi^{(e)}(x) = \Lambda^{(\ell)}\Psi_D(x) = \left(\Lambda^{(\ell)}\Pi_{iR}\Psi_D\right) = \begin{pmatrix} e & \mu & \tau & \cdot \end{pmatrix}_R. \quad (73)$$

In terms of the chiral triplet fields in (65) and (66), the Lagrangian density of the kinetic part including the gauge interactions of the fundamental fermions is given in the generic form

$$\mathcal{L}_{kg} = \bar{\Psi}_L(x)\Gamma^\mu\mathcal{D}_\mu\Psi_L(x) + \bar{\Psi}_U(x)\Gamma^\mu\mathcal{D}_\mu\Psi_U(x) + \bar{\Psi}_D(x)\Gamma^\mu\mathcal{D}_\mu\Psi_D(x) \quad (74)$$

with the covariant derivatives \mathcal{D}_μ which act on the triplet fields as follows:

$$\mathcal{D}_\mu\Psi_L(x) = \left(\partial_\mu - gA_\mu^a(x)\frac{1}{2}\tau_a - g'B_\mu(x)\frac{1}{2}Y - g_cA_{c\mu}^a(x)\frac{1}{2}\lambda_a - g_X X_{c\mu}^a(x)\frac{1}{2}Q_X\right)\Psi_L(x), \quad (75)$$

$$\mathcal{D}_\mu\Psi_U(x) = \left(\partial_\mu - g'B_\mu(x)\frac{1}{2}Y - g_cA_{c\mu}^a(x)\frac{1}{2}\lambda_a - g_X X_{c\mu}^a(x)\frac{1}{2}Q_X\right)\Psi_U(x) \quad (76)$$

and

$$\mathcal{D}_\mu\Psi_D(x) = \left(\partial_\mu - g'B_\mu(x)\frac{1}{2}Y - g_cA_{c\mu}^a(x)\frac{1}{2}\lambda_a - g_X X_{c\mu}^a(x)\frac{1}{2}Q_X\right)\Psi_D(x) \quad (77)$$

where $A_\mu^a(x)$, $B_\mu(x)$, $A_{c\mu}^a(x)$ and $X_\mu(x)$ are respectively the gauge fields for the electroweak isospin, hypercharge $Y = y + Q_{B-L}$, color symmetry and $U_X(1)$ symmetry. This density \mathcal{L}_{kg} is readily proved to reproduce the gauge interactions of the standard model which includes also the right-handed neutrino species and the fourth additional family of fermions with X charge.

In the present scheme, the chiral triplet fields include the repetitional degrees of freedom for family structure. This feature enables us to formulate the Yukawa interactions in an intrinsic mechanism without bringing in so many unknown parameters from outside. In terms of chiral triplet fields and the Higgs field, the Lagrangian density of the Yukawa interactions can be expressed to be

$$\begin{aligned} \mathcal{L}_Y &= g_u\bar{\Psi}^{(q)}(x)\mathcal{Y}_u\tilde{\phi}(x)\Psi^{(u)}(x) + g_d\bar{\Psi}^{(q)}(x)\mathcal{Y}_d\phi(x)\Psi^{(d)}(x) \\ &+ g_\nu\bar{\Psi}^{(\ell)}(x)\mathcal{Y}_\nu\tilde{\phi}(x)\Psi^{(\nu)}(x) + g_e\bar{\Psi}^{(\ell)}(x)\mathcal{Y}_e\phi(x)\Psi^{(e)}(x) + \text{h.c.} \end{aligned} \quad (78)$$

where the kernel operator \mathcal{Y}_a ($a = u, d, \nu, e$) works to induce mixings among repetitional family modes. To be commutative with the generators of color symmetry, the operator must be composed solely of the elements of the family algebra A_f . The quark part of the density \mathcal{L}_Y is decomposed into the Yukawa interactions among component quark fields as

$$g_a \bar{\Psi}^{(q)} \mathcal{Y}_a \Psi^{(a)} = g_a \sum_{ij} \bar{\Psi}^{(q)} \Pi_{iR} \mathcal{Y}_a \Pi_{jR} \Psi^{(a)} = \sum_{ij} \bar{\Psi}_i^{(q)} [g_a \langle\langle iR | \mathcal{Y}_a | jR \rangle\rangle] \Psi_j^{(a)} \quad (79)$$

where $a = u, d$. This result and the similar decomposition of the lepton part lead us to the interpretation that the quantities

$$g_a \langle\langle i | \mathcal{Y}_a | j \rangle\rangle \equiv g_a \langle\langle iR | \mathcal{Y}_a | jR \rangle\rangle \quad (80)$$

are the Yukawa coupling constants of the a sector ($a = u, d, \nu, e$).

Let us express the kernel operators \mathcal{Y}_a generically by \mathcal{Y} . In general, \mathcal{Y} can consist of arbitrary elements of the trio and solo algebras $A_{(t)}$ and $A_{(s)}$. At present we have no principle to restrict its form. It is experimental information in flavor physics which work to restrict the composition of the kernel operator. Namely, we must repeat trial and error of data fittings with candidates of kernel operator whose compositions are fixed under possible working hypotheses. Here, we restrict our consideration to the ordinary trio families and impose one simple hypothesis that the kernel is self-adjoint and composed of the following components as

$$\mathcal{Y} = \mathcal{Y}_{(t) sym} + \mathcal{Y}_{(t) asym} \quad (81)$$

where $\mathcal{Y}_{(t) sym}$ and $\mathcal{Y}_{(t) asym}$ are, respectively, composed of such elements of $A_{(t)}$ that are symmetric and asymmetric parts under the cyclic permutation of the S_3 . The symmetric and asymmetric parts are determined to be

$$\begin{aligned} \mathcal{Y}_{(t) sym} &= 2a(\pi_2 - \pi_5 + \pi_7) + 2b(\pi_1 + \pi_4 + \pi_6) + c\Pi_{(t)} \\ &= a[\sigma_1 \otimes \sigma_2 \otimes \sigma_3 + \sigma_3 \otimes \sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1 \otimes \sigma_3 \\ &\quad - \sigma_3 \otimes \sigma_2 \otimes \sigma_1 - \sigma_1 \otimes \sigma_3 \otimes \sigma_2 - \sigma_2 \otimes \sigma_1 \otimes \sigma_3] \\ &+ b[1 \otimes \sigma_1 \otimes \sigma_1 + \sigma_1 \otimes 1 \otimes \sigma_1 + \sigma_1 \otimes \sigma_1 \otimes 1 \\ &\quad + 1 \otimes \sigma_2 \otimes \sigma_2 + \sigma_2 \otimes 1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_2 \otimes 1] + c\Pi_{(t)} \end{aligned} \quad (82)$$

and

$$\begin{aligned} \mathcal{Y}_{(t) asym} &= e(\pi_1 - \pi_6) \\ &= e[\sigma_1 \otimes \sigma_1 \otimes 1 + \sigma_2 \otimes \sigma_2 \otimes 1 - 1 \otimes \sigma_1 \otimes \sigma_1 - 1 \otimes \sigma_2 \otimes \sigma_2]. \end{aligned} \quad (83)$$

All coefficients in these operators are real constants. The forms of the operators $\mathcal{Y}_{(t) sym}$ in (82) are unique. We choose the operator $\mathcal{Y}_{(t) asym}$ in (83) from many other possibility so as to simplify numerical analyses of Dirac mass matrices.

To calculate the Yukawa coupling constants as the matrix elements in (80), let us specify matrix representations of the chiral projection operators in A_γ as

$$\ell = \frac{1}{2}(1 - \gamma_5) = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad r = \frac{1}{2}(1 + \gamma_5) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (84)$$

The eigenvectors of these operators are given in the forms

$$|\ell\rangle\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} |\psi\rangle \\ -|\psi\rangle \end{pmatrix} e^{i\theta}, \quad |r\rangle\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} |\psi\rangle \\ |\psi\rangle \end{pmatrix} \quad (85)$$

where $|\psi\rangle$ is a normalized Pauli spinor and θ is an arbitrary phase. In terms of the ordered products of these component eigenvector, we are able to construct the state vectors of interaction modes $|j\rangle = |jR\rangle$ as follows:

$$|1\rangle = |r\ell\ell\rangle, \quad |2\rangle = |\ell r\ell\rangle, \quad |3\rangle = |\ell\ell r\rangle, \quad |4\rangle = |rrr\rangle \quad (86)$$

with $|r\ell\ell\rangle = |r\rangle_1 |\ell\rangle_2 |\ell\rangle_3$ etc. Then, careful calculation of the matrix elements of the kernel operator in (81) results in

$$\begin{aligned} (\langle i | \mathcal{Y} | j \rangle) &\propto \left(\begin{array}{c|cccc} & |r\ell\ell\rangle & |\ell r\ell\rangle & |\ell\ell r\rangle & |rrr\rangle \\ \hline \langle r\ell\ell| & & & & \\ \langle \ell r\ell| & & \text{Kernel} & & \\ \langle \ell\ell r| & & \mathcal{Y} & & \\ \langle rrr| & & & & \end{array} \right) \\ &= \begin{pmatrix} c & (ia+b+e)e^{i\alpha_3} & (-ia+b)e^{i\alpha_2} & 0 \\ (-ia+b+e)e^{-i\alpha_3} & c & (ia+b-e)e^{i\alpha_1} & 0 \\ (ia+b)e^{-i\alpha_2} & (-ia+b-e)e^{-i\alpha_1} & c & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (87)$$

where $\alpha_j = \theta_{j+1} - \theta_{j+2}$ ($\theta_{j+3} = \theta_j$) is the difference of the phases in the component ket vectors $|\ell\rangle_{j+1}$ and $|\ell\rangle_{j+2}$ ($|\ell\rangle_{j+3} = |\ell\rangle_j$). Due to the orthogonality between the trio and solo algebras, there exist no Yukawa coupling constant which works to mix the ordinary trio and additional solo families of fundamental fermions.

In the present theory of fundamental fermions, we inherit the Weinberg-Salam mechanism for the breakdown of the electroweak symmetry. At and below the energy scale where the Higgs field takes the vacuum expectation value, the Yukawa interaction (78) results in the Dirac mass matrices. For the charged sector of the trio families, we obtain, generically, the following Hermitian matrix.

$$\mathcal{M} \propto P^\dagger \begin{pmatrix} c & ia+b+e & -ia+b \\ -ia+b+e & c & ia+b-e \\ ia+b & -ia+b-e & c \end{pmatrix} P \quad (88)$$

where

$$P = \text{diagonal} \left(e^{-i\theta_1}, e^{-i\theta_2}, e^{-i\theta_3} \right) \quad (89)$$

is the diagonal phase matrix. The charged solo fermion in the isolated fourth mode has the mass proportional to the constant f .

7. Discussion

In this way, we have formulated an algebraic theory for description of external and internal attributes and symmetries of fundamental fermions. The basic ingredients of the theory are the *triplet algebra* consisting of all the TDPs of the Dirac matrices and the *triplet field* forming the multiple spinor representation of the Lorentz group.

The triplet algebra is decomposed into the product of the external and internal algebras. All elements of the external algebra A_{ex} are invariant under the action of permutation group S_3 which works to exchange the order of the A_γ elements in the TDP. The internal algebra is the product

of the subalgebras of the family and color degrees of freedom, A_f and A_c . Both subalgebras have the trio plus solo conformation as $A_f = \{L, R\}\{A_{(t)} \cup A_{(s)}\}$ and $A_c = A^{(q)} \cup A^{(\ell)}$. All of the trio and solo subalgebras, $A_{(t)}$, $A^{(q)}$, $A_{(s)}$ and $A^{(\ell)}$, are irreducible under the group S_3 . On these algebraic constructions, the triplet field possesses the internal group structure of the color symmetry and the $B - L$ and X charges.

The Weinberg-Salam theory of electroweak interaction is formulated in the triplet field formalism by postulating the existence of a set of the triplet fields of the electroweak isospin doublet and singlets as in (65) and (66). It should be noticed that the y charge has only a temporal role to distinguish the singlet fields Ψ_U and Ψ_D and do not have its own gauge field. In the Lagrangian density in (74), the ordinary gauge field $B_\mu(x)$ of $U_Y(1)$ symmetry is assumed to couple to the hypercharge $Y = y + Q_{B-L}$. It is tempting to develop a left and right symmetric extension of electroweak interaction and to relate the y charge to a component of a right-handed isospin. However, we leave the study of such possibility in future as an open problem.

Most characteristic aspect of the triplet field formalism lies in the points that fundamental fermions are identified with component fields of the triplet fields and interactions among the fermions can be interpreted as transitions among component fields. The gauge and Yukawa interactions of the standard model are arranged, respectively, in the Lagrangian densities \mathcal{L}_{kg} in (74) and \mathcal{L}_Y in (78) with diagonal and off-diagonal combinations of chiral triplet fields. The trio plus solo formations of family modes enables us to separate the interactions of ordinary fermions and those of the additional fourth fermions. Using the kernel in (81), we are able to *calculate* the Yukawa coupling constants as its matrix elements. Note that the phase of CP violation originates from the phases in the eigenvectors in (85).

The mass matrix of Dirac type in (88) has new and unique structure. If the parameters b and c take close and dominant values, it becomes the well known mass matrix of democratic type. With this mass matrix, we can carry out numerical analyses on the mass spectra and the weak mixing matrix for the ordinary quark sector and succeed to explain all data within experimental accuracy. Results of numerical analyses will be published elsewhere.

For further investigations of the lepton sector $\Psi^{(\ell)} = \Psi_{(t)}^{(\ell)} + \Psi_{(s)}^{(\ell)}$ and the solo family sector $\Psi_{(s)} = \Psi_{(s)}^{(q)} + \Psi_{(s)}^{(\ell)}$, it is necessary to enrich contents of bosonic fields. So far, only one kind of Higgs field $\phi(x)$ of the standard model in (67) is chosen to describe the Weinberg-Salam mechanism. The main reason for this restriction is to severely suppress the appearance of the flavor-changing-neutral-current. To explain the smallness of the neutrino masses, however, it is required to introduce a different type of scalar field which brings about the seesaw mechanism by creating the so-called Majorana masses. Further, we must introduce an extra Higgs field which works to break down the $U_X(1)$ symmetry and metamorphose the gauge field $X_\mu(x)$ to a vector field with a sufficiently large mass. This Higgs field is assumed to couple solely to the solo family so that the fourth fermions acquire sufficiently heavy masses compared with the ordinary trio fermions. We can expect that physics of the solo family sector can naturally be related to rapidly developing LHC experiments and astrophysics. There exist plausible candidates for the dark matter in the hitherto-unobserved solo family sector. Such possibilities must be pursued in the triplet field formalism.

Last but not least. In this article, we have identified the internal algebra A_{in} with the centralizer of the algebra for the Lorentz transformation in (10) as

$$A_{in} = C(\{\Sigma_{\mu\nu} : A_T\}) = \langle 1 \otimes \gamma_\mu \otimes \gamma_\mu, \gamma_\mu \otimes 1 \otimes \gamma_\mu, \gamma_5 \otimes \gamma_5 \otimes \gamma_5 \rangle.$$

This choice turns out convenient to find the algebra of the family degrees of freedom through the decompositions of the chiral projection operators in (13) and (14) which are impossible in the external algebra A_{ex} . However, once we found the projection operators Π_j to the four family modes and the relations in (20) which allow to regenerate the chiral operators Π_{jh} , it is possible

to use the centralizer of the external algebra

$$A_{in} = C(\{A_{ex} : A_T\}) = \langle 1 \otimes \gamma_\mu \otimes \gamma_\mu, \gamma_\mu \otimes 1 \otimes \gamma_\mu \rangle \quad (90)$$

as an alternative definition of the internal algebra. Mathematically, this simple choice which satisfies the relation

$$A_T = A_{ex}A_{in}, \quad A_{ex} \cap A_{in} = \emptyset \quad (91)$$

seems relevant since it is faithful to the Coleman-Mandula theorem[3, 4] insisting that the “spacetime and internal symmetries cannot be combined in any but a trivial way”. However, both algebras have no essential difference in physics.

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